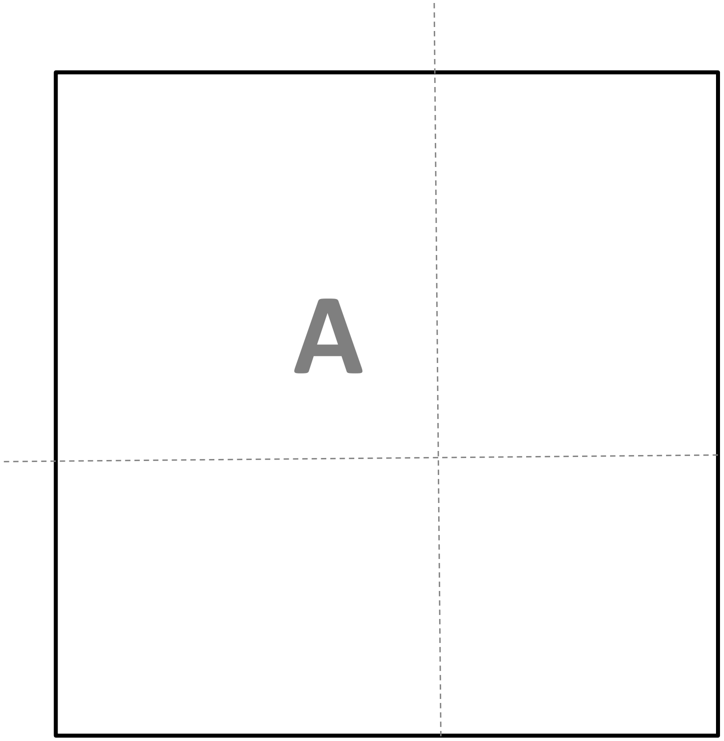
Strategies for inversion of the additive relationship matrix among genotyped animals

P. Faux and N. Gengler

University of Liege, Gembloux Agro-Bio Tech, Animal Science Unit

Introduction: The case of A_{22} vs. A

✓ $\mathbf{A}_{22}$ = subpart of **A** whose inversion is required, e.g. in ssGBLUP



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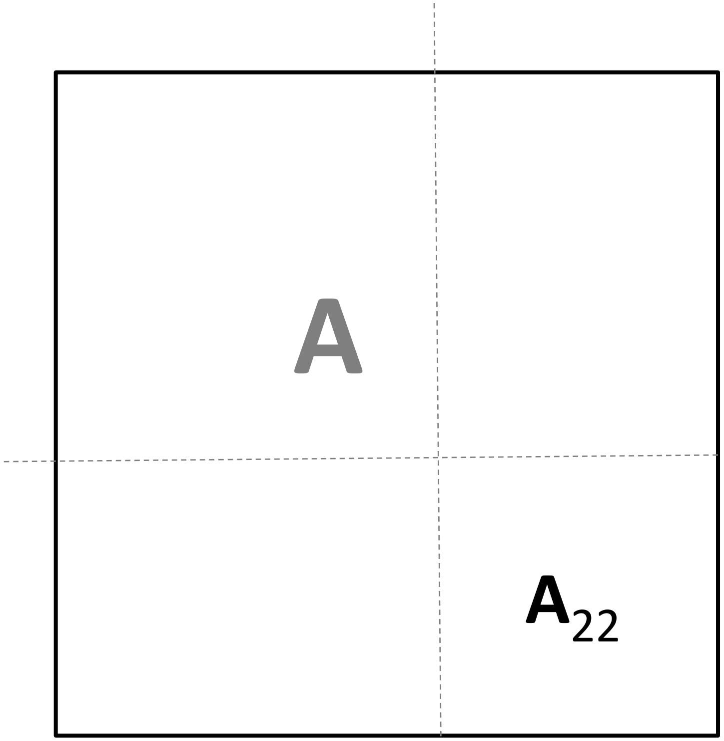

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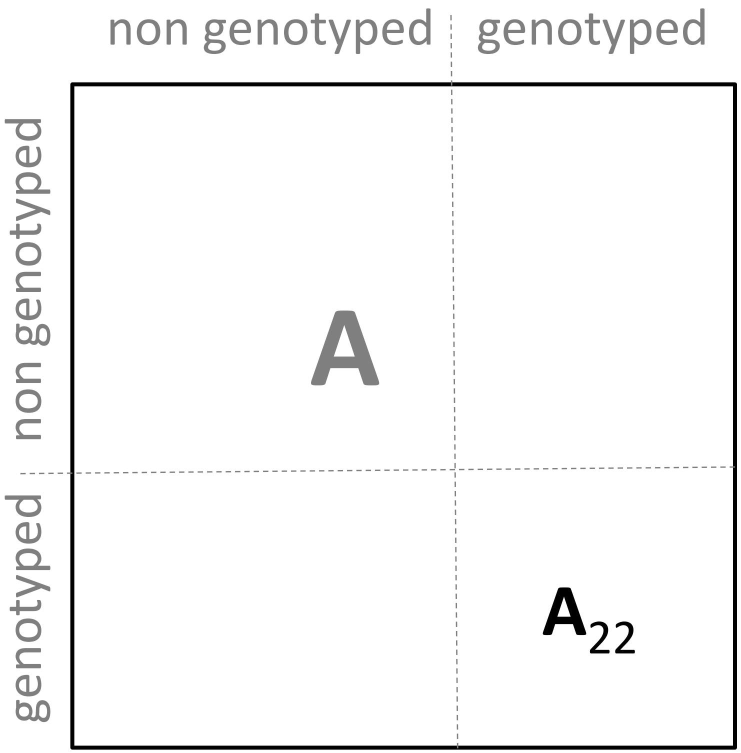

Introduction: The case of A_{22} vs. A

✓The inverse of **A** is computed as **a sum of vector products** (Henderson, 1976)

$$\mathbf{A}_{(i)}^{-1} = \begin{bmatrix} \mathbf{A}_{(i-1)}^{-1} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + \alpha_{(i)} \begin{bmatrix} -\mathbf{b}_{(i)} \\ 1 \end{bmatrix} \begin{bmatrix} -\mathbf{b}'_{(i)} & 1 \end{bmatrix}$$

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$$\mathbf{A}_{(i)}^{-1} = \left(\mathbf{T}_{A(i)}^{-1}\right)' \mathbf{D}_{A(i)}^{-1} \mathbf{T}_{A(i)}^{-1}$$

$$\mathbf{T}_{A(i)}^{-1} = \begin{bmatrix} \mathbf{T}_{A(i-1)}^{-1} & \mathbf{0} \\ -\mathbf{b}'_{(i)} & \mathbf{1} \end{bmatrix}$$

$$\mathbf{D}_{A(i)}^{-1} = \begin{bmatrix} \mathbf{D}_{A(i-1)}^{-1} & \mathbf{0} \\ \mathbf{0}' & \alpha_{(i)} \end{bmatrix}$$

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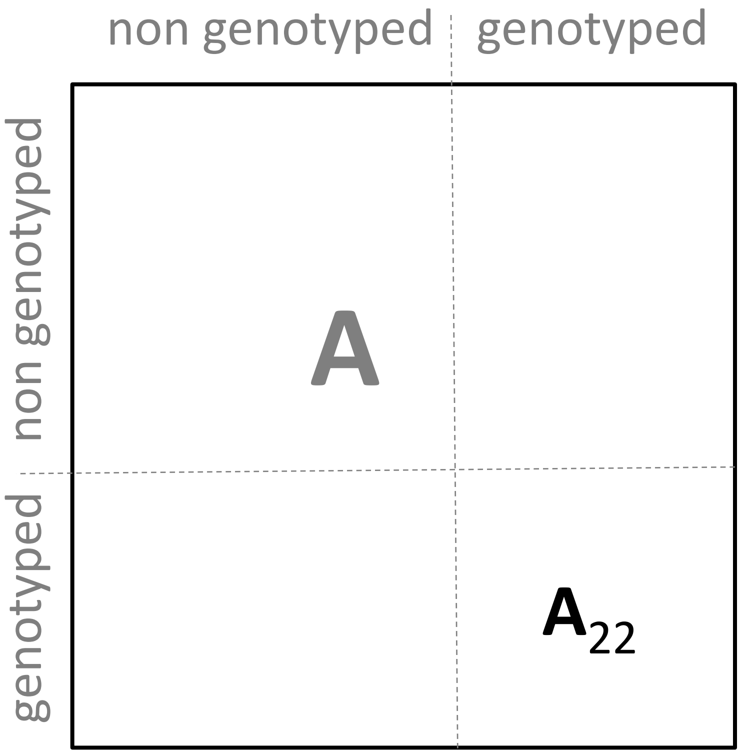
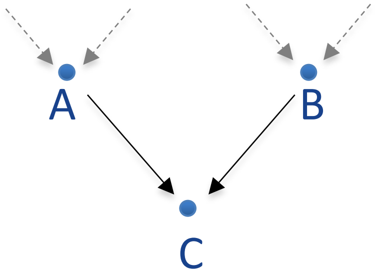


Sparsity in the inverse factor of A_{22}

✓**Example**: An animal and its parents



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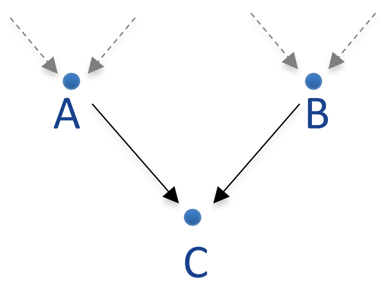
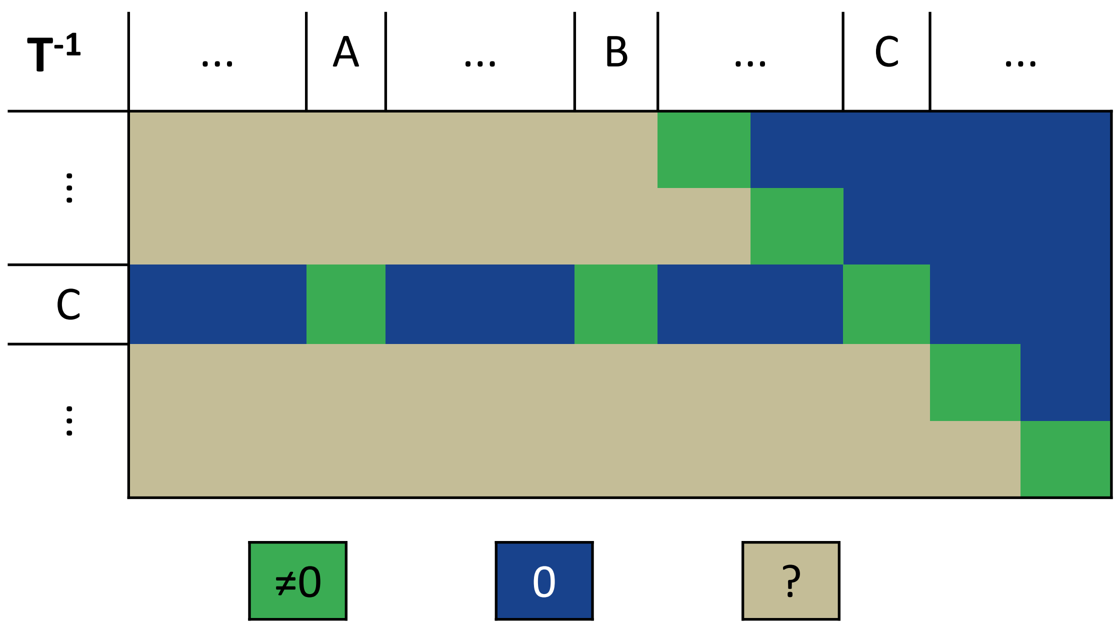




Issues and Objective

✓How sparse is the inverse of $\mathbf{A}_{22}$?

... How sparse is the inverse factor ($\mathbf{T}^{-1}$) of $\mathbf{A}_{22}$?

✓How a putative sparsity could be used in computation of the inverse?

→ Main objective: **To avoid useless computations**

Sparsity in the inverse factor of A_{22}

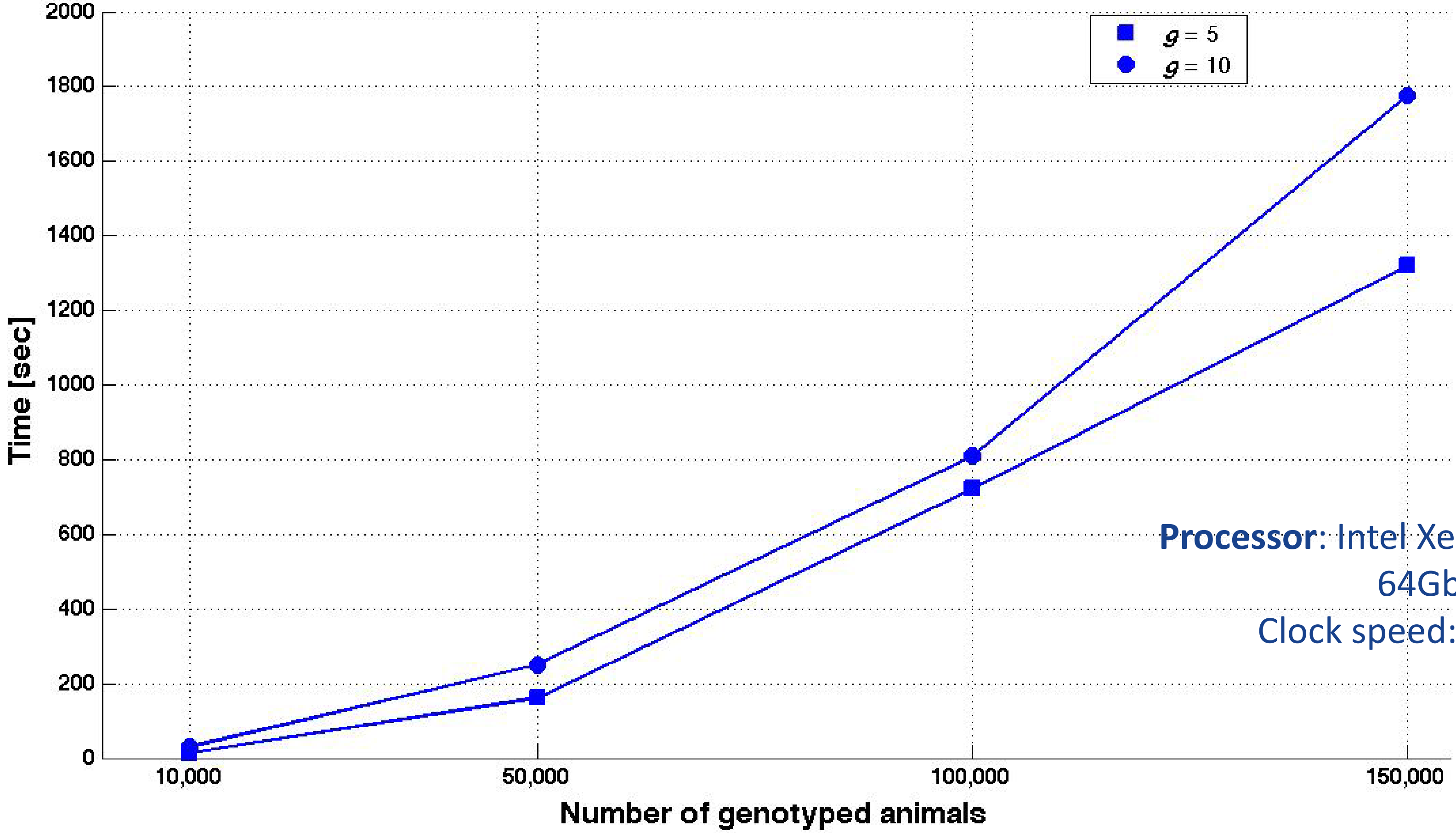
✓How to deal with **more complex cases**?

✓By a **comprehensive search in the pedigree**

- ✓« SP Algorithm »
- ✓Explores pedigree brances and apply simple rules
- ✓Uses only pedigree and incidence vector
- ✓Returns a symbolic inverse factorization

Sparsity in the inverse factor of A_{22}

✓Some **performances** on different sizes of $\mathbf{A}_{22}$:



Processor: Intel Xeon 64-bit
64Gb of RAM,
Clock speed: 3.16GHz

Strategies to take sparsity into account

1. Successive construction of the inverse

$$\mathbf{A}_{22(i)}^{-1} = \begin{bmatrix} \mathbf{A}_{22(i-1)}^{-1} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + \alpha_{(i)} \begin{bmatrix} -\mathbf{b}_{(i)} \\ 1 \end{bmatrix} \begin{bmatrix} -\mathbf{b}'_{(i)} & 1 \end{bmatrix}$$

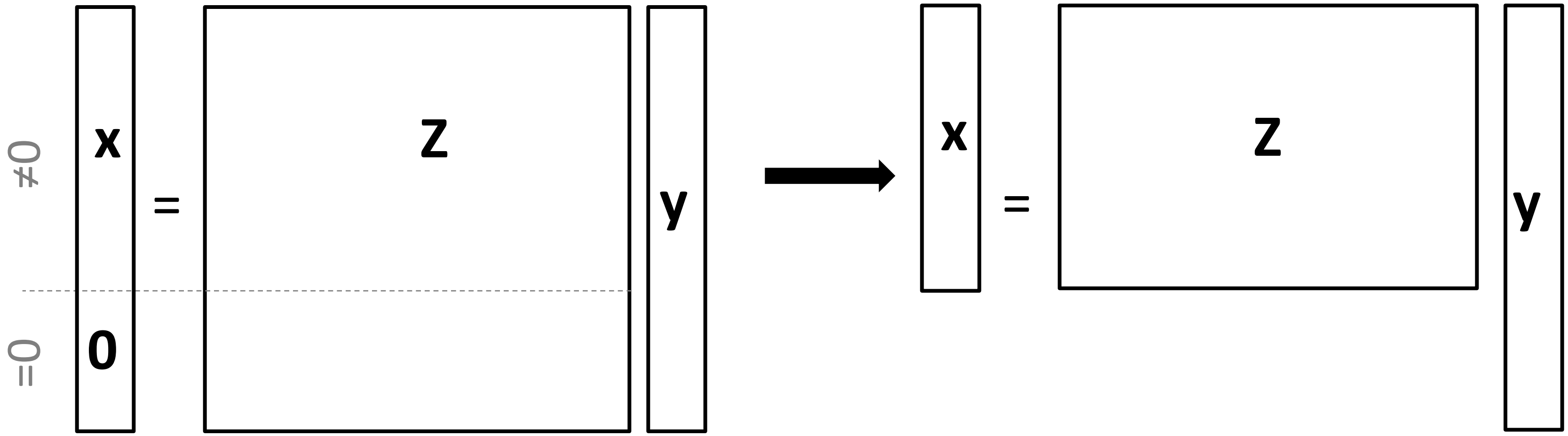
How to get **b**?

1. $\mathbf{b}_{(i)} = \mathbf{A}_{22(i-1)}^{-1} \mathbf{A}_{22(i-1)}(:, 1 : i-1)$
2. $\mathbf{A}_{22(i-1)} \mathbf{b}_{(i)} = \mathbf{A}_{22(i-1)}(:, 1 : i-1)$

Strategies to take sparsity into account

1. Restricting the product only to elements of **b** different from 0

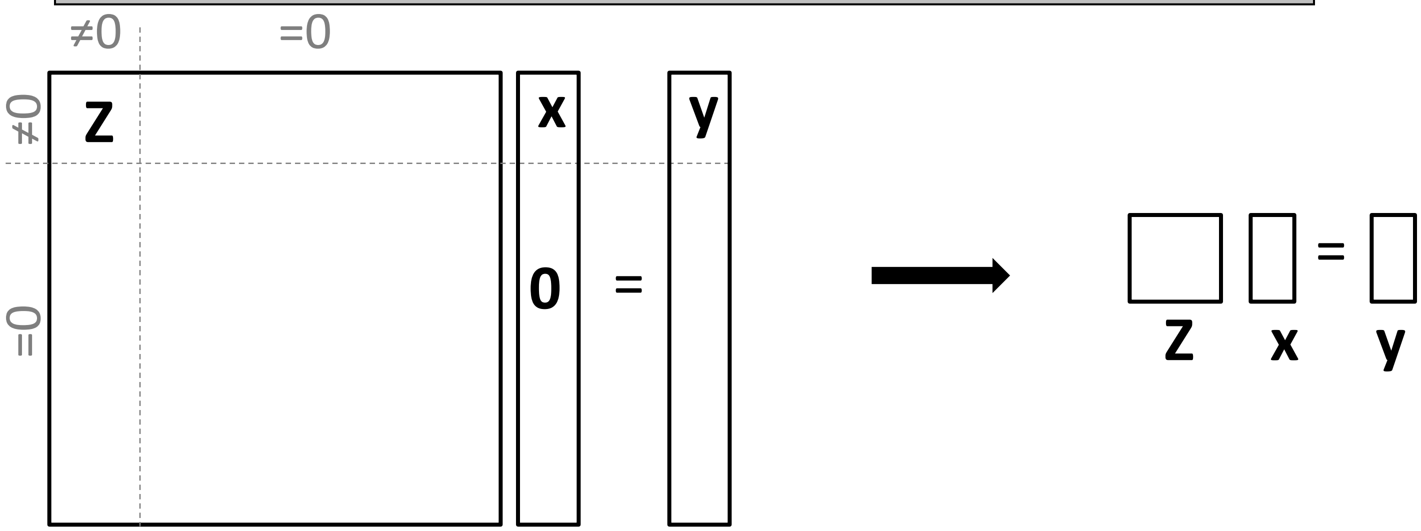
$$\mathbf{b}_{(i)} = \mathbf{A}_{22(i-1)}^{-1}\mathbf{A}_{22(i-1)}(:,1:i-1) \quad \rightarrow \quad \mathbf{x} = \mathbf{Z}\mathbf{y}$$



Strategies to take sparsity into account

2. Solving a linear system of lower size

$$\mathbf{A}_{22(i-1)}\mathbf{b}_{(i)} = \mathbf{A}_{22(i-1)}(:,1:i-1) \quad \rightarrow \quad \mathbf{Zx} = \mathbf{y}$$



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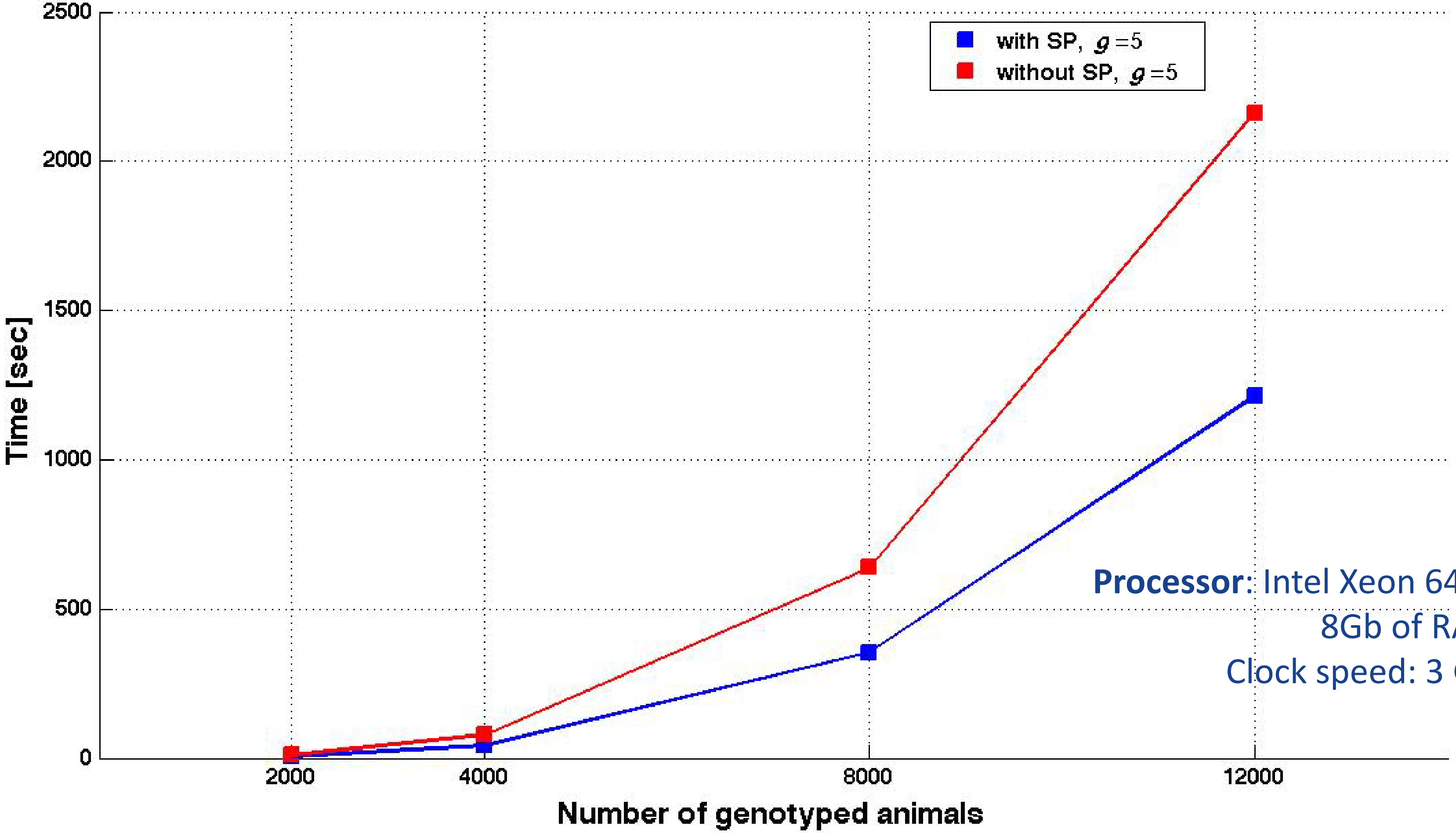

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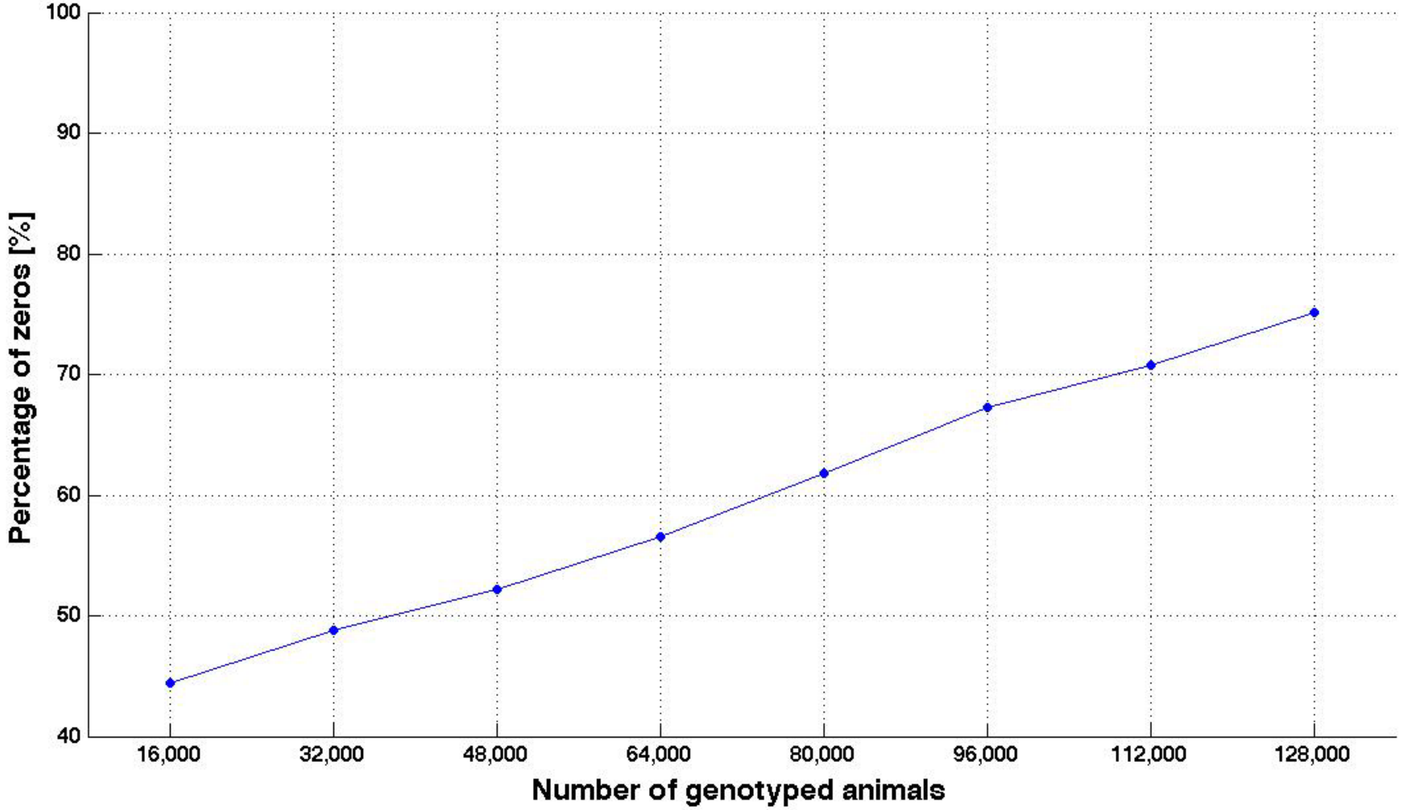
Strategies to take sparsity into account

- ✓Order of $\mathbf{A}_{22}$ = Number of genotyped animals
  - ✓Depends on the pedigree (depth, lines, ...)



Strategies to take sparsity into account

3. Storing the inverse of $\mathbf{A}_{22}$ from time to time and updating this inverse only for recent animals

$$\mathbf{A}_{22(t+1)}^{-1} = \begin{bmatrix} \mathbf{A}_{22(t)}^{-1} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + \alpha_{(x)} \begin{bmatrix} -\mathbf{b}_{(x)} \\ 1 \end{bmatrix} \begin{bmatrix} -\mathbf{b}'_{(x)} & 1 \end{bmatrix}$$

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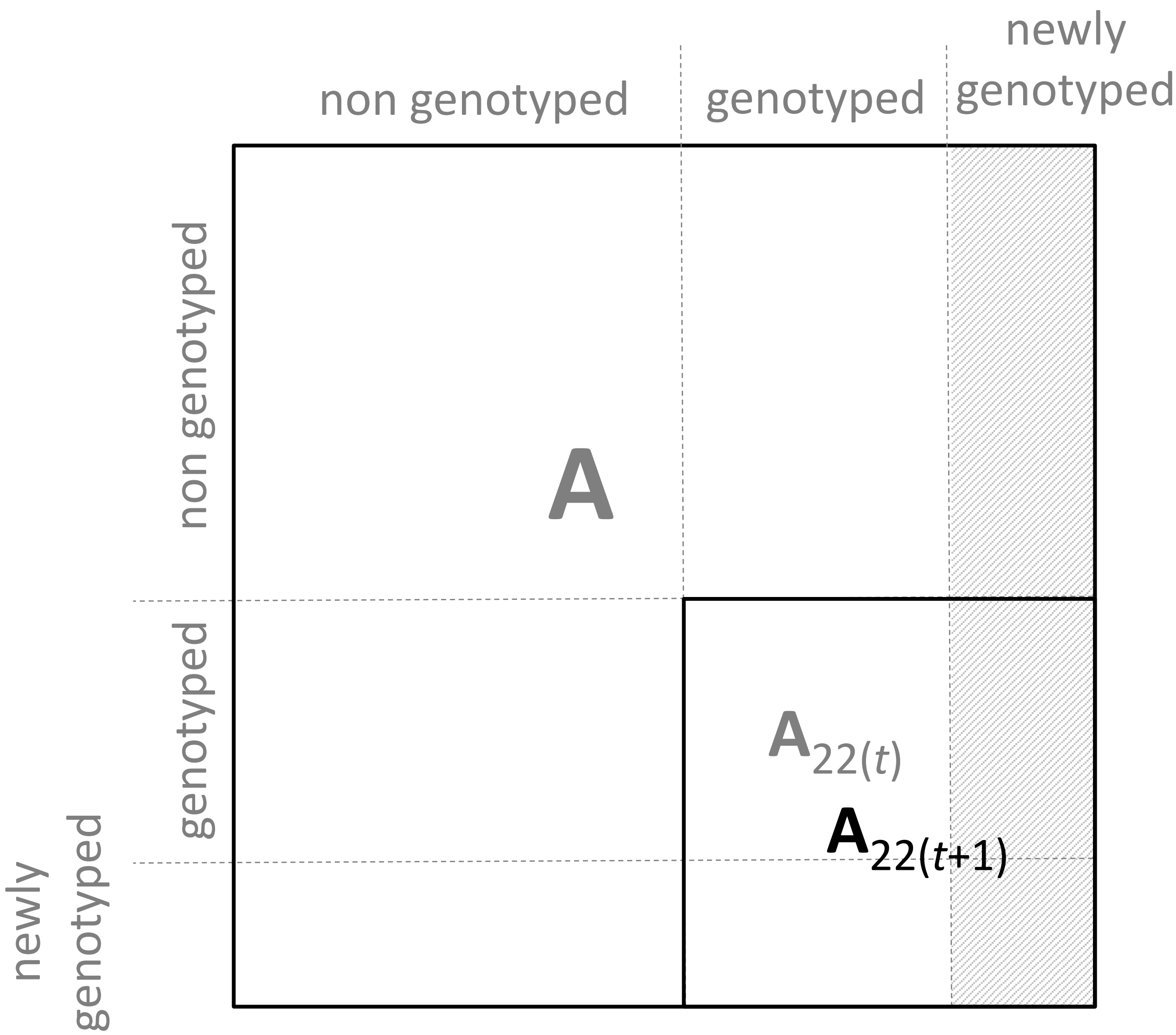

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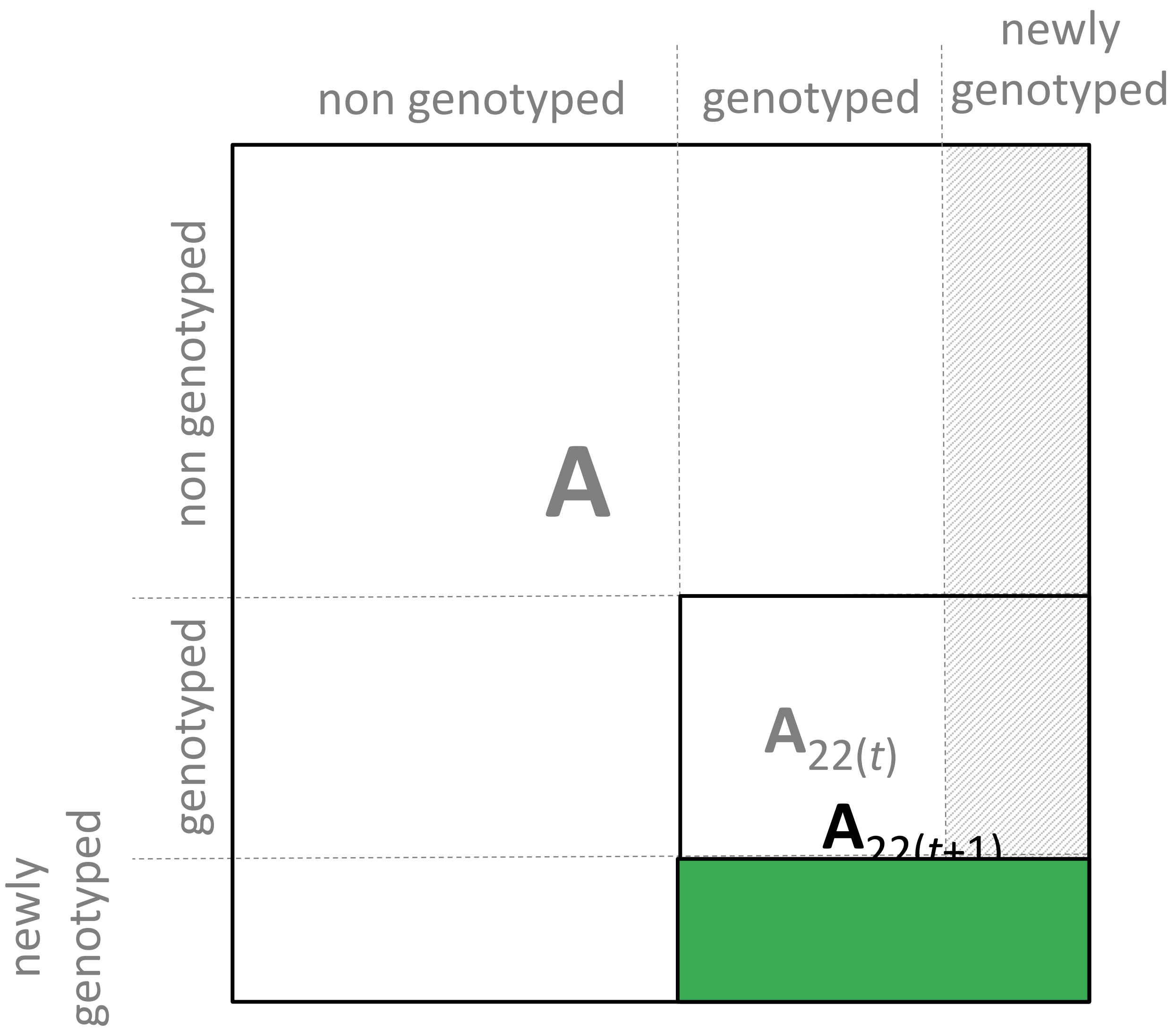

Take-home messages

1. Sparsity pattern of the inverse of $\mathbf{A}_{22}$ can be set up without matrix computations, even for large matrices

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2. Using sparsity reduces time for inversion, if that inversion uses the inverse factor

3. As the order of $\mathbf{A}_{22}$ increases, inversion shrinks to solve multiple small linear systems that are identified by SP algorithm

Acknowledgements

- Fonds National de la Recherche Luxembourg (FNR)
- CONVIS s.c.
- Association Wallonne de l’Elevage (AWE)
- S. Vanderick, F. Colinet (Gembloux Agro-Bio Tech)